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ON THE MULTISET-REPRESENTABLE GRAPHS

Jihoon Choi

Abstract. A graph is said to be $[[k]]$-*representable* if we can assign a multiset on $\{1, 2, \ldots, k\}$ to each vertex such that two vertices are adjacent if and only if their corresponding multisets are comparable under set inclusion. A graph is said to be *multiset-representable* if it is $[[k]]$-representable for some positive integer k. In this paper, we study the basic properties of multiset-representability.

1. Introduction

All the terms used in this paper can be found in [1]. For simplicity, we denote the set $\{1, 2, \ldots, k\}$ by $[k]$. The sets of positive integers and nonnegative integers are denoted by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. A *multiset* on $[k]$ is a function $m : [k] \to \mathbb{N}_0$, which is also called a *multiplicity function*. For example, the function $m : [4] \to \mathbb{N}_0$ defined by $m(1) = 3$, $m(2) = 0$, $m(3) = 1$, and $m(4) = 2$ is a multiset on $[4]$, which can be represented by $\{1, 1, 1, 3, 4, 4\}$, or simply as 111344. The multiset whose values are all zero is called the *emptyset* and is denoted by $\emptyset$.

Let $[[k]]$ denote the family of multisets on $[k]$. We define an irreflexive relation $\prec$ on $[[k]]$ such that $m_1 \prec m_2$ if and only if $m_1(x) \le m_2(x)$ for each $x \in [k]$. For example, $\{1, 3\} \prec \{1, 3, 4\} \prec \{1, 1, 3, 4, 4, 4\}$. Note that $\{1, 3, 4\} \not\prec \{1, 3, 4\}$ since $\prec$ is defined to be irreflexive. It is easy to see that $\prec$ is a strict partial order on $[[k]]$.

Let G be a simple graph. We say that G is $[[k]]$-*representable* if we can injectively assign a multiset m_v on $[k]$ to each vertex v such that xy is an edge in G if and only if $m_x \prec m_y$ or $m_y \prec m_x$. In other words, G is $[[k]]$-representable if G is the comparability graph of $(V(G), \prec)$, where $V(G)$ is a family of multisets on $[k]$. Such an assignment on $V(G)$ is called a $[[k]]$-*assignment*. The smallest positive integer k such that G

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is $[[k]]$-representable is called *the multiset-index of* G and is denoted by $i_m(G)$. If there is no such integer k, we define $i_m(G) = \infty$. If G is $[[k]]$-representable for some positive integer k, or equivalently, if $i_m(G) < \infty$, then G is said to be *multiset-representable*.

We present two examples. The trivial graph K_1 with vertex v is $[[1]]$-representable by assigning $\emptyset$ or $\{1\}$ to v. Therefore, $i_m(K_1) = 1$. The path graph $P_4 = v_1v_2v_3v_4$ is $[[2]]$-representable by the multiset assignment $v_1 = \{1\}, v_2 = \{1, 2\}, v_3 = \{2\}$, and $v_4 = \{2, 2\}$, thus $i_m(P_4) \leq 2$.

2. The $[[k]]$-representability of graphs

Every multiset-representable graph is a comparability graph. Therefore, multiset-representable graphs are always transitively orientable. In fact, the following theorem holds:

Theorem 2.1. *A graph is multiset-representable if and only if it is transitively orientable.*

Proof. It is sufficient to prove the "if" part. We will prove the stronger statement: If a graph G on n vertices has a transitive orientation D, then there exists an $[[n]]$-assignment $\phi : V(G) \to [[n]]$ such that $(x, y) \in A(D)$ if and only if $\phi(y) \prec \phi(x)$.

We proceed by induction on n. If $n = 1$, then G is $[[1]]$-representable by assigning $\{1\}$ to the single vertex of G. Assume that the statement holds for $n \geq 1$. Now, consider a graph G with $n + 1$ vertices and a transitive orientation D. By transitivity, D has a vertex u with an indegree of 0. Then $D - u$ is a transitive orientation of $G - u$. By the induction hypothesis, there exists an $[[n]]$-assignment $\phi : V(G - u) \to [[n]]$ such that $(x, y) \in A(D - u)$ if and only if $\phi(y) \prec \phi(x)$. To extend ϕ to $V(G)$, we define $\phi(u)$ as follows:

$$\phi(u)(i) = \begin{cases} \max\{\phi(v)(i) \mid v \in N_D^+(u)\}, & \text{for } 1 \leq i \leq n; \\ 1, & \text{for } i = n + 1. \end{cases}$$

It is easy to check that the extended ϕ is the desired $[[n + 1]]$-assignment of G. □

Odd cycles of length at least 5 are not comparability graphs. Therefore, they are examples of non-multiset-representable graphs. The next proposition shows that $[[k]]$-representability is hereditary.

Proposition 2.2. *Every induced subgraph of a* $[[k]]$*-representable graph is also* $[[k]]$*-representable.*

Proof. Suppose G has a $[[k]]$-assignment ϕ. Take an induced subgraph H of G. Then the restriction of ϕ to $V(H)$ naturally yields a $[[k]]$-assignment of H. Hence, H is $[[k]]$-representable. □

An *apex* of a graph is a vertex that is adjacent to all the other vertices.

PROPOSITION 2.3. *Let G be a $[[k]]$-representable graph with $k \geq 2$. Then, adding an isolated vertex or an apex still results in a $[[k]]$-representable graph.*

Proof. Let ϕ be a $[[k]]$-assignment of G. We denote $\phi(v)$ by ϕ_v for each $v \in V(G)$. We define a new multiplicity function $\phi'_v : [k] \to \mathbb{N}_0$ such that $\phi'_v(1) = \phi_v(1) + 2$ and $\phi'_v(i) = \phi_v(i)$ for $i \neq 1$. Clearly, the map ϕ' defined by $\phi'(v) = \phi'_v$ remains a valid $[[k]]$-assignment of G.

Let $V(G) = \{v_1, \ldots, v_r\}$. If we add a new vertex w with $\phi'(w) = \{1\}$, then $w \prec v_i$ for each $i \in [r]$. Thus, w is comparable with every other vertex because each vertex $v_1, \ldots, v_r$ contains at least two 1's in its $[[k]]$-assignment under ϕ'. Therefore, G together with an apex w remains $[[k]]$-representable.

Let $M = \max\{\phi'_v(2) \mid v \in V(G)\}$. If we add a new vertex u with $\phi'(u) = \{1, 2, 2, \ldots, 2\}$ (one copy of 1 and $M+1$ copies of 2), then none of the vertices $v_1, \ldots, v_r$ is comparable with u since u contains fewer 1's and more 2's than the other vertices. Therefore, G together with an isolated vertex u remains $[[k]]$-representable. □

We can characterize $[[k]]$-representable graphs from a geometric point of view, which will be useful in computing the multiset-index. We write $\mathbb{N}_0^k = \mathbb{N}_0 \times \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ (k times). For two k-tuples $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{N}_0^k$, we write $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for each $i = 1, \ldots, k$.

THEOREM 2.4. *A graph G is $[[k]]$-representable if and only if there exists an injective function $f : V(G) \to \mathbb{N}_0^k$ such that, for any distinct vertices x and y, the following property holds:*

($\star$) *$xy \in E(G)$ if and only if $f(x) \leq f(y)$ or $f(y) \leq f(x)$.*

Proof. Suppose G is a $[[k]]$-representable graph with a $[[k]]$-assignment ϕ. Denote $\phi(v)$ by ϕ_v for each $v \in V(G)$. We define $f : V(G) \to \mathbb{N}_0^k$ by $f(v) = (\phi_v(1), \phi_v(2), \ldots, \phi_v(k))$. Since ϕ is injective, f is also injective. Moreover, f satisfies the property ($\star$) by the definition.

Conversely, suppose there exists an injective function $f : V(G) \to \mathbb{N}_0^k$ that satisfies the property ($\star$). For each $v \in V(G)$, we define the multiplicity function $\phi_v : [k] \to \mathbb{N}_0$ by $(\phi_v(1), \phi_v(2), \ldots, \phi_v(k)) = f(v)$. Then the function ϕ that maps $v \in V(G)$ to ϕ_v is a $[[k]]$-assignment of G. □

Theorem 2.4 tells us that a given graph G is $[[2]]$-representable if and only if we can place the vertices of G as lattice points in the plane $\mathbb{R}^2$, where adjacent vertices are connected by a line with a non-positive slope (i.e., negative, zero, or undefined slope).

The *union* of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph defined by $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by adding all the edges connecting a vertex in G_1 to a vertex in G_2.

PROPOSITION 2.5. *Assume that $k \geq 2$. Then the join of two $[[k]]$-representable graphs is also $[[k]]$-representable. Moreover, the union of two $[[k]]$-representable graphs is also $[[k]]$-representable.*

Proof. For convenience, we adopt the following notation: for a function $f : [k] \to \mathbb{N}_0$ and a nonnegative integer M, let $f + M : [k] \to \mathbb{N}_0$ denote the function defined by $(f + M)(v) = f(v) + M$.

Let G_1 and G_2 be $[[k]]$-representable graphs with $[[k]]$-assignments ϕ_1 and ϕ_2, respectively. Define:

- $M = 1 + \max \{\phi_1(v)(i) \mid v \in V(G_1), i = 1, \ldots, k\}$
- $M_1 = 1 + \max \{\phi_1(v)(1) \mid v \in V(G_1)\}$
- $M_2 = 1 + \max \{\phi_2(v)(2) \mid v \in V(G_2)\}$

To show that $G_1 \vee G_2$ is $[[k]]$-representable, we define a function $\psi : V(G_1) \cup V(G_2) \to [[k]]$ by:

- For each $v_1 \in V(G_1)$, $\psi(v_1) = \phi_1(v_1)$.
- For each $v_2 \in V(G_2)$, $\psi(v_2) = \phi_2(v_2) + M$.

This construction ensures that for all $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, we have $\psi(v_1) \prec \psi(v_2)$, because the entries in $\psi(v_2)$ are uniformly larger than those in $\psi(v_1)$. Additionally, the comparability condition is preserved within each $V(G_1)$ and $V(G_2)$ due to the properties of ϕ_1 and ϕ_2. Therefore, ψ is a valid $[[k]]$-assignment for $G_1 \vee G_2$, showing that the join is $[[k]]$-representable.

To show that $G_1 \cup G_2$ is $[[k]]$-representable, we define a function $\eta : V(G_1) \cup V(G_2) \to [[k]]$ as follows:

- For each $v_1 \in V(G_1)$, $\eta(v_1)(i) = \begin{cases} \phi_1(v_1)(i) + M_2, & \text{if } i = 2; \\ \phi_1(v_1)(i), & \text{otherwise.} \end{cases}$
- For each $v_2 \in V(G_2)$, $\eta(v_2)(i) = \begin{cases} \phi_2(v_2)(i) + M_1, & \text{if } i = 1; \\ \phi_2(v_2)(i), & \text{otherwise.} \end{cases}$

This construction ensures that within $V(G_1)$ and $V(G_2)$, comparability is preserved, as the modified entries do not interfere with the comparability conditions. Moreover, no edges can form between G_1 and G_2 because all vertices in G_1 have fewer 1's and more 2's than any vertex in G_2. Hence, η is a $[[k]]$-assignment of $G_1 \cup G_2$, proving that the union is also $[[k]]$-representable. □

Corollary 2.6. *Let G be a graph, and let $G_1, \ldots, G_r$ be its components. Then $i_m(G) = \max\{i_m(G_i) \mid i = 1, \ldots, r\}$.*

Proof. Since G is the union of $G_1, \ldots, G_r$, the result follows directly from Proposition 2.5. □

Corollary 2.6 tells us that it is sufficient to study the multiset-indices of connected graphs.

3. On the multiset-indices of graphs

In this section, we investigate the multiset-indices of some graphs.

Proposition 3.1. *A graph has multiset-index 1 if and only if it is a complete graph.*

Proof. Suppose that a graph G has multiset-index 1. Then G has a $[[1]]$-assignment. This means that every vertex is assigned a multiset consisting solely of copies of 1, so every pair of vertices is comparable. Thus, G is complete.

Conversely, suppose that G is a complete graph with vertices $v_1, \ldots, v_n$. By assigning a multiset consisting of i copies of 1 to each vertex v_i, we obtain a $[[1]]$-assignment of G. Hence, G has multiset-index 1. □

Proposition 3.2. *Every edgeless graph with at least two vertices has multiset-index 2.*

Proof. Let G be an edgeless graph with vertices $v_1, \ldots, v_n$ for some $n \geq 2$. For each vertex v_i, we define a multiplicity function $m_{v_i} : [k] \to \mathbb{N}_0$ by $m_{v_i}(1) = i - 1$ and $m_{v_i}(2) = n - i + 1$. Then no two of $v_1, \ldots, v_n$ are comparable and therefore, G is $[[2]]$-representable. Since G has at least two vertices, G is not complete and so it is not $[[1]]$-representable by Proposition 3.1. Hence $i_m(G) = 2$. □

Proposition 3.3. *Every complete bipartite graph $K_{m,n}$ has the multiset-index:*

$$i_m(K_{m,n}) = \begin{cases} 1, & \text{if } m = n = 1 \\ 2, & \text{otherwise.} \end{cases}$$

Proof. If $m = 1$ and $n = 1$, then $K_{m,n} = K_2$ and so $i_m(K_{m,n}) = 1$ by proposition 3.1. Assume $m > 1$ or $n > 1$. Then $i_m(K_{m,n}) > 2$ by the same proposition. For each positive integer r, let I_r denote the edgeless graph on r vertices Then $K_{m,n}$ is isomorphic to $I_m \vee I_n$. By Propositions 2.5 and 3.2, $i_m(K_{m,n}) = i_m(I_m \vee I_n) \leq 2$. Hence $i_m(K_{m,n}) = 2$. □

PROPOSITION 3.4. *The path P_n on n vertices ($n \geq 2$) has multiset-index* 2.

Proof. We denote $P_n = v_1 v_2 \cdots v_n$ and define a function $\phi : V(P_n) \to [[2]]$ as follows: If i is odd, then $\phi(v_i)$ a multiset on [2] consisting of $\lceil \frac{n}{2} \rceil - \frac{i+1}{2}$ 1's and $\frac{i-1}{2}$ 2's; if i is even, then $\phi(v_i)$ is a multiset on [2] consisting of $\lceil \frac{n}{2} \rceil - \frac{i}{2}$ 1's and $\frac{i}{2}$ 2's. For example, when $n = 5$, the function $\phi : V(P_5) \to [[2]]$ is given by $\phi(v_1) = 11$, $\phi(v_2) = 112$, $\phi(v_3) = 12$, $\phi(v_4) = 122$, $\phi(v_5) = 22$. Briefly speaking, the multisets $\phi(v_1), \phi(v_2), \ldots, \phi(v_n)$ are obtained one by one by adding 2, deleting 1, adding 2, deleting 1, and so on. By the construction, it is easy to see that ϕ gives a [[2]]-assignment of P_n. □

Consider the cycle $C_n = v_1 v_2 \cdots v_n v_1$. Since $C_3 = K_3$, we have $i_m(C_3) = 1$. When $n = 4$, the function $\phi : \{v_1, v_2, v_3, v_4\} \to [[2]]$ defined by $\phi(v_1) = 1$, $\phi(v_2) = 112$, $\phi(v_3) = 2$, $\phi(v_4) = 122$ is a [[2]]-assignment of C_4. Therefore $i_m(C_4) = 2$.

THEOREM 3.5. *For any $n \geq 5$, C_n is not* [[2]]*-representable.*

Proof. If n is odd, then C_n is not transitively orientable, so it cannot be [[2]]-representable. Now, suppose C_{2n} is [[2]]-representable for some integer $n \geq 3$. Let $\phi : V(C_{2n}) \to [[2]]$ be a [[2]]-assignment. Since C_{2n} is triangle-free, neither $\phi(v_{i-1}) \prec \phi(v_i) \prec \phi(v_{i+1})$ nor $\phi(v_{i-1}) \succ \phi(v_i) \succ \phi(v_{i+1})$ can occur in C_{2n}. Therefore, for each i, either $\phi(v_{i-1}) \prec \phi(v_i) \succ \phi(v_{i+1})$ or $\phi(v_{i-1}) \succ \phi(v_i) \prec \phi(v_{i+1})$ must hold. Hence, we may assume:

$$\phi(v_1) \prec \phi(v_2) \succ \phi(v_3) \prec \phi(v_4) \succ \cdots \prec \phi(v_{2n}) \succ \phi(v_1).$$

Note that $\phi(v_1), \phi(v_3), \phi(v_5), \ldots, \phi(v_{2n-1})$ are pairwise incomparable. Among them, we may assume that $\phi(v_3)$ has the fewest 1's.

Case 1: $\phi(v_5)(1) \leq \phi(v_1)(1)$. In this case, we have $\phi(v_3)(1) \leq \phi(v_5)(1) \leq \phi(v_1)(1)$ and $\phi(v_3)(2) \geq \phi(v_5)(2) \geq \phi(v_1)(2)$. Since $\phi(v_2)$ is comparable with both $\phi(v_1)$ and $\phi(v_3)$, either $\phi(v_2) \prec \phi(v_1)$ and $\phi(v_2) \prec \phi(v_3)$, or $\phi(v_1) \prec \phi(v_2)$ and $\phi(v_3) \prec \phi(v_2)$ must hold. In either case, $\phi(v_2)$ must also be comparable with $\phi(v_5)$, which leads to a contradiction.

Case 2: $\phi(v_5)(1) \geq \phi(v_1)(1)$. In this case, we reverse the roles of v_1 and v_3 from Case 1 and proceed with a parallel argument to show that $\phi(v_4)$ must be comparable with $\phi(v_1)$, which again leads to a contradiction.

Therefore, C_{2n} is not [[2]]-representable for any integer $n \geq 3$. $\square$

4. Concluding remarks

It would be an interesting problem to characterize [[k]]-representable graph for a given integer k. In addition, we hope to find a graph with arbitrarily large multiset-index.

References

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Jihoon Choi
Department of Mathematics Education
Cheongju University
Cheongju, Chungbuk 28503, Republic of Korea
E-mail: jihoon@cju.ac.kr